The extended Amato index and its application to income data

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The research proposes an alternative inequality index, an extension of the Amato index, which we term the extended Amato index. The original Amato index, which represents the length of the Lorenz curve and forms part of the inequality zone's perimeter, serves as the foundation for this extension. The authors derive this index by computing the ratio of the inequality zone's perimeter to the inequality triangle's perimeter, both of which emerge from the egalitarian line and the Lorenz curve. The study apply the extended Amato index to empirical and Lorenz function formulations, using data on income employment per household from The Ghana Living Standards Survey IV. The results suggest that the extended Amato index fulfils all properties of the inequality measure, except egalitarian zero. However, the authors rectify this by performing minimum-maximum scaling adjustments on the extended Amato index, yielding the adjusted extended Amato index, which satisfies all properties of the inequality measure, including egalitarian zero. The empirical findings reveal high levels of income inequality in Ghana in 1998, as indicated by the value of the extended Amato index. Furthermore, when the value of the extended Amato index is calculated using the Lorenz function formulation, the accurate specification of the Lorenz function is validated due to its strong alignment with the empirical Lorenz curve. Ultimately, these findings can guide policies aimed at reducing inequality through wealth redistribution.

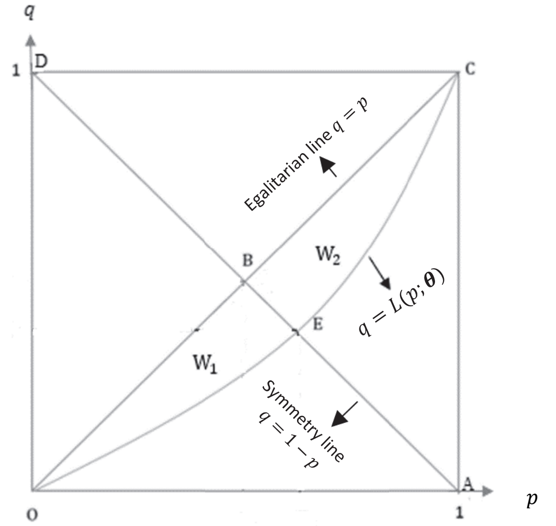
Introduction

The Lorenz curve, introduced by Lorenz (1905), provides a graphical depiction of wealth distribution inequality (Kocziszky et al. 2018, Kashour 2023), and is frequently employed to visually represent income inequality. Furthermore, the Lorenz curve is applied extensively in various research fields to analyse disparities, including in public health (Chang–Halfon 1997, Kobayashi–Takaki 1992), public transport (Delbosc–Currie 2011, Raza et al. 2023), medical research (Lee 1999), social welfare analysis (Foster–Shorrocks 1988, Sitthiyot–Holasut 2020, Slottje 1989, Wang et al. 2009), industry (Smith 1947), ecology and bionomics (Damgaard–Weiner 2000, Harvey et al. 2011), agricultural analysis (Sadras–Bongiovanni 2004), hydrology (Lu et al. 2021), and residential energy use (Szép et al. 2022).

The Lorenz curve serves as a useful tool for analysing the income distribution within and between diverse income groups and evaluating income inequality within a specific region. The Lorenz curve, represented by line $\overline{\mathrm{OEC}}$, is shown in Figure 1. Mathematically, the Lorenz curve is a function that depicts the relationship between the cumulative proportion of income $(q;\ 0 \le q \le 1)$ and that of households $(p;\ 0 \le p \le 1)$. The functional form of the Lorenz curve with the parameter vector $\boldsymbol{\theta}$ is defined by $L(p;\boldsymbol{\theta})$ (Aggarwal 1984, Basmann et al. 1990, Chotikapanich 1993, Gupta 1984, Hossain–Saeki 2003, Kakwani 1980, Kakwani–Podder 1973, Ortega et al. 1991, Paul–Shankar 2020, Rasche et al. 1980, Rohde 2009, Sarabia 1997, Sarabia et al. 2001, 2010, 2015, Schader–Schmid 1994, Wang et al. 2009).

Figure 1

Visualization of the Lorenz curve and inequality zone



In Figure 1, the horizontal axis p represents the cumulative proportion of unit households, and the vertical axis q represents the cumulative proportion of income received by the cumulative proportion of households. The egalitarian line $\overline{\mathrm{OBC}}$ serves as a reference line indicating perfect equality, where the cumulative proportion of households is equal to that of the income $(q = p)$. This condition indicates a scenario in which household income is distributed equally, with each household receiving the same income. The zone W $(\overline{\mathrm{OBCE}})$ is defined by the egalitarian line and the Lorenz curve. Note that within this zone, two subzones emerge: subzone W_1 $(\overline{\mathrm{OBE}})$, and subzone W_2 $(\overline{\mathrm{BEC}})$, with the line of symmetry serving as the boundary between them.

The zone $\overline{\mathrm{OBCE}}$ represents the segment of society in which income inequality is present. An expansion in this area signifies a rise in income inequality, whereas a contraction indicates a decrease in inequality. Consequently, the area of the inequality zone is incorporated into the computation of the Gini index, which is calculated as the ratio of the area of $\overline{\mathrm{OBCE}}$ to the area of the triangle $\overline{\mathrm{OAC}}$ (Sitthiyot–Holasut 2020).

In this study, we propose that in addition to the area of the zone $\overline{\mathrm{OBCE}}$, we can also use its perimeter to measure income inequality. Note that as the inequality zone expands, both its area and perimeter increase. This suggests that the Lorenz curve deviates further from the egalitarian line, signifying a higher level of income inequality. Conversely, if the inequality zone contracts, both the area and perimeter decrease, indicating that the Lorenz curve approaches the egalitarian line, which reflects a lower level of income inequality.

Note too that the length of the Lorenz curve is one of the components contributing to the perimeter of the inequality zone $\overline{\mathrm{OBCE}}$, and is also an Amato index, as described by Amato (1968) and Arnold (2012). Compared with other indices, such as the Pietra index (Pietra 1915), Kolkata index (Banerjee et al. 2019, 2020, Chatterjee et al. 2017), and Zenga index (Greselin et al. 2010, Langel–Tillé 2012, Zenga 2007), the Amato index provides a more comprehensive representation of inequality. Furthermore, it can be associated with the asymmetry of the Lorenz curve, with the Amato index demonstrating consistency with the size of the inequality zone $\overline{\mathrm{OBCE}}$.

Therefore, this study proposes an alternative measure of inequality that incorporates the perimeter of the inequality zone $\overline{\mathrm{OBCE}}$ and that of the triangle $\overline{\mathrm{OAC}}$ (the inequality triangle), represented as the ratio of the perimeter of $\overline{\mathrm{OBCE}}$ to that of $\overline{\mathrm{OAC}}$. We refer to the proposed measure of inequality as the extended Amato index, given that the Amato index represents the length of the Lorenz curve, which is a component of the perimeter of $\overline{\mathrm{OBCE}}$, as discussed earlier. The extended Amato index demonstrates consistency with the size of the inequality zone $\overline{\mathrm{OBCE}}$. When the size of $\overline{\mathrm{OBCE}}$ increases, so does its perimeter, resulting in a higher extended Amato index. Similarly, a reduction in the size of $\overline{\mathrm{OBCE}}$ leads to a decrease in its perimeter and in the extended Amato index.

Basic theory

Lorenz function

Suppose the function $L(p;\boldsymbol{\theta})$ has a parameter vector θ and is continuous in the interval [0,1]. Then, the function is a Lorenz function if it satisfies the following criteria (Hossain–Saeki 2003, Paul–Shankar 2020, Rohde 2009, Sarabia 1997, Sarabia et al. 2010, 2015):

$$L(p=0;\boldsymbol{\theta})=0; \quad L(p=1;\boldsymbol{\theta})=1; \; L'(p;\boldsymbol{\theta})\geq 0; \; L''(p;\boldsymbol{\theta})\geq 0,$$

where $L'(p;\boldsymbol{\theta})$ and $L''(p;\boldsymbol{\theta})$ are the first and second derivatives, respectively, of $L(p;\boldsymbol{\theta})$ with respect to p. Accordingly, the criteria $L'(p;\boldsymbol{\theta})\geq 0$ and $L''(p;\boldsymbol{\theta})\geq 0$ guarantee that the Lorenz function $L(p;\boldsymbol{\theta})$ is convex in the interval $0\leq p\leq 1$. This convexity aptly describes how disparities in an income distribution affect the egalitarian line.

Proportion cumulative

The initial and crucial step in estimating the parameters of the Lorenz function involves constructing p_i and q_i based on the observation order index and on household income $x_j, j=1,\dots,\mathcal{N}$ ($\mathcal{N}$ is the number of sample households), respectively, as follows:

a) Arrange the data set as presented in Table 1.

Table 1

Layout data set

Sample households, *j*	Household income
1st household	x_1
2nd household	x_2
⋮	⋮
$\mathcal{N}$th household	$x_{\mathcal{N}}$

b) Then, sort the x_j variables in ascending order, implying that households adjust to their household income, as presented in Table 2.

Table 2

Sorted results based on the data set in Table 1

Ordered sample households *(j)*	Ordered household income $x_{(j)}$
1st household	x_1
2nd household	x_2
⋮	⋮
$\mathcal{N}$th household	$x_{\mathcal{N}}$

c) Construct p_i and q_i using the cumulative proportion from Table 2, as follows:

Table 3

Data $\boldsymbol{p_i}$ and $\boldsymbol{q_i}$

p_i	q_i
$p_0 = 0$	$q_0 = (x_{(0)})/\sum_{i=0}^{\mathcal{N}} x_{(i)} = 0, x_{(0)} = 0$
$p_1 = (0+1)/\sum_{i=0}^{\mathcal{N}} i$	$q_1 = (x_{(0)} + x_{(1)})/\sum_{i=0}^{\mathcal{N}} x_{(i)}$
$p_2 = (0+1+2)/\sum_{i=0}^{\mathcal{N}} i$	$q_2 = (x_{(0)} + x_{(1)} + x_{(2)})/\sum_{i=0}^{\mathcal{N}} x_{(i)}$
$\vdots$	$\vdots$
$p_i = (0+1+2+\cdots+i)/\sum_{i=0}^{\mathcal{N}} i$	$q_i = (x_{(0)} + x_{(1)} + x_{(2)} + \cdots + x_{(i)})/\sum_{i=0}^{\mathcal{N}} x_{(i)}$
$\vdots$	$\vdots$
$p_{\mathcal{N}} = \frac{\sum_{i=0}^{\mathcal{N}} i}{\sum_{i=0}^{\mathcal{N}} i} = 1$	$q_{\mathcal{N}} = \frac{\sum_{i=0}^{\mathcal{N}} x_{(i)}}{\sum_{i=0}^{\mathcal{N}} x_{(i)}} = 1$

The nonlinear least squares method using the Levenberg–Marquardt algorithm

In this study, the parameter vector $\boldsymbol{\theta}$ of the Lorenz function $L(p;\boldsymbol{\theta})$ is estimated using the nonlinear least squares estimation method. Accordingly, an estimate of $\boldsymbol{\theta}$ is obtained using parameter estimation in $L(p;\boldsymbol{\theta})$ by minimizing $S^*(\boldsymbol{\theta})$:

$$\begin{aligned}\min_{\boldsymbol{\theta}} S^*(\boldsymbol{\theta}) &= \min_{\boldsymbol{\theta}} \frac{1}{2}\big(\boldsymbol{q}^* - L(\boldsymbol{p};\boldsymbol{\theta})\big)^{\mathrm{T}}\big(\boldsymbol{q}^* - L(\boldsymbol{p};\boldsymbol{\theta})\big) \\ &= \min_{\boldsymbol{\theta}} \frac{1}{2}\boldsymbol{f}(\boldsymbol{\theta})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}) \\ &= \min_{\boldsymbol{\theta}} \frac{1}{2}\sum_{i=0}^{\mathcal{N}}\big(f_i(\boldsymbol{\theta})\big)^2,\end{aligned}$$

where $\boldsymbol{q}^* = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{\mathcal{N}} \end{bmatrix}, \boldsymbol{p} = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_{\mathcal{N}} \end{bmatrix}, \boldsymbol{\theta} = \begin{bmatrix} \theta_1 \\ \theta_j \\ \vdots \\ \theta_d \end{bmatrix}$, and d is number of parameters.

Eq. (9) adopts the idea of the distance between the empirical and theoretical distribution functions used in the Kolmogorov–Smirnov test. This distance is inferred between $\boldsymbol{q}^*$, representing the cumulative proportion of income obtained empirically from the data, and $L(\boldsymbol{p};\boldsymbol{\theta})$, indicating the Lorenz function. The Lorenz function is also a nonlinear function, which makes the objective function $S^*(\boldsymbol{\theta})$ intricate and dependent on the specification of the functional form of $L(\boldsymbol{p};\boldsymbol{\theta})$. Consequently, solving the minimization process of the objective function $S^*(\boldsymbol{\theta})$ analytically is difficult.

$$S^*(\boldsymbol{\theta}) = \frac{1}{2}\sum_{i=0}^{\mathcal{N}}\big(q_i - L(p_i;\boldsymbol{\theta})\big)^2.$$

The first derivative $S^*(\boldsymbol{\theta})$ with respect to p_i, is

$$\frac{dS^*(\boldsymbol{\theta})}{dp_i} = -\sum_{i=0}^{\mathcal{N}} \big(q_i - L(p_i;\boldsymbol{\theta})\big)L'(p_i;\boldsymbol{\theta})\,.$$

The second derivative $S^*(\boldsymbol{\theta})$ with respect to p_i is

$$\frac{d^2S^*(\boldsymbol{\theta})}{dp_i^2} = \sum_{i=0}^{\mathcal{N}} \Big(\big(L'(p_i;\boldsymbol{\theta})\big)^2 - \big(q_i - L(p_i;\boldsymbol{\theta})\big)L''(p_i;\boldsymbol{\theta})\Big).$$

The objective function $S^*(\boldsymbol{\theta})$ is a convex function if $\frac{dS^*(\boldsymbol{\theta})}{dp_i} \geq 0$ and $\frac{d^2S^*(\boldsymbol{\theta})}{dp_i^2} \geq 0$, depending on the specification of the functions $L'(p;\boldsymbol{\theta})$ and $L''(p;\boldsymbol{\theta})$. This rule implies that the objective function $S^*(\boldsymbol{\theta})$ is not necessarily a convex function. As a result, we require an iteration process to obtain a convergent estimated value of $\boldsymbol{\theta}$. In addition, it is crucial to acknowledge that during the estimation process, we may encounter a local minimum for $S^*(\boldsymbol{\theta})$. The estimator $\widehat{\boldsymbol{\theta}}$ represents a local minimizer: $S^*\big(\widehat{\boldsymbol{\theta}}\big) \leq S^*(\boldsymbol{\theta})$ for $\big\|\boldsymbol{\theta} - \widehat{\boldsymbol{\theta}}\,\big\| \leq \epsilon$, where ϵ is positive and very small.

In each iteration of $L(\boldsymbol{p};\boldsymbol{\theta})$, the parameter vector $\boldsymbol{\theta}$ is updated with $\boldsymbol{\theta} + \boldsymbol{\delta}$. To determine $\boldsymbol{\delta}$, the function $L(\boldsymbol{p},\boldsymbol{\theta} + \boldsymbol{\delta})$ is approximated by linearization, as follows:

$$L(\boldsymbol{p};\boldsymbol{\theta} + \boldsymbol{\delta}) \approx L(\boldsymbol{p};\boldsymbol{\theta}) + \boldsymbol{J}^*\boldsymbol{\delta}, \qquad \boldsymbol{J}^* = \frac{\partial L(\boldsymbol{p};\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}.$$

Thus,

$$S^*(\boldsymbol{\theta} + \boldsymbol{\delta}) \approx \frac{1}{2}(\boldsymbol{q}^* - L(\boldsymbol{p};\boldsymbol{\theta}) - \boldsymbol{J}^*\boldsymbol{\delta})^{\mathrm{T}}(\boldsymbol{q}^* - L(\boldsymbol{p};\boldsymbol{\theta}) - \boldsymbol{J}^*\boldsymbol{\delta})$$

$$= \frac{1}{2}\boldsymbol{f}(\boldsymbol{\theta})^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}) + \boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}) + \frac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{\delta}$$

$$\mathcal{S}(\boldsymbol{\delta}) = S^*(\boldsymbol{\theta}) + \boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}) + \tfrac{1}{2}\boldsymbol{\delta}^{\mathrm{T}}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{\delta}, \tag{1}$$

where $-\boldsymbol{J}^* = \boldsymbol{J}, \qquad \boldsymbol{J} = \frac{\partial \boldsymbol{f}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}, \boldsymbol{J} \in \mathbb{R}^{\mathcal{N}\times d}$.

In Eq. (1), we determine the first derivative of $\boldsymbol{\delta}$, yielding

$$\mathcal{S}'(\boldsymbol{\delta}) = S^{*\prime}(\boldsymbol{\theta} + \boldsymbol{\delta}) = \boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}) + \boldsymbol{J}^{\mathrm{T}}\boldsymbol{J}\boldsymbol{\delta}. \tag{2}$$

The Gauss–Newton step $\boldsymbol{\delta}_{\mathrm{gn}}$ obtained from the first-order condition in Eq. (2) becomes

$$\boldsymbol{\delta}_{\mathrm{gn}} = -(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J})^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}), \tag{3}$$

where $\boldsymbol{J}$ is a full-rank matrix. In Eq. (3), we modify the component $(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J})$ by adding the damping parameter λ in the component, yielding:

$$\boldsymbol{\delta}_{\mathrm{lm}} = -(\boldsymbol{J}^{\mathrm{T}}\boldsymbol{J} + \lambda\mathbf{I}_d)^{-1}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}), \lambda \geq 0, \tag{4}$$

where $\boldsymbol{\delta}_{\mathrm{lm}}$ is the Levenberg–Marquardt step, and $\mathbf{I}_d$ is the identity matrix of dimension $d \times d$. If $\lambda = 0$, Eq. (4) becomes a Gauss–Newton step, if λ is close to zero, Eq. (4) tends to be a Gauss–Newton step, and if λ is very large, Eq. (4) becomes

$$\boldsymbol{\delta}_{\mathrm{sd}} \cong -\frac{1}{\lambda}\boldsymbol{J}^{\mathrm{T}}\boldsymbol{f}(\boldsymbol{\theta}). \tag{5}$$

Eq. (5) is the steepest descent step. This equation shows that the damping parameter λ greatly affects the direction and size of the step; as a result, we perform an iteration process to obtain the estimator $\widehat{\boldsymbol{\theta}}$. Furthermore, note that the parameter λ can be updated during the iteration process, and is controlled by the gain ratio (Madsen et al. 2004),

$$\Phi = \frac{S^*(\boldsymbol{\theta}) - S^*(\boldsymbol{\theta} + \boldsymbol{\delta}_{\text{lm}})}{\frac{1}{2}\boldsymbol{\delta}_{\text{lm}}^{\text{T}}\left(\lambda\boldsymbol{\delta}_{\text{lm}} - \boldsymbol{J}^{\text{T}}\boldsymbol{f}(\boldsymbol{\theta})\right)}. \tag{6}$$

When the value of Φ is large in Eq. (6), then $\mathcal{S}(\boldsymbol{\delta}_{\text{lm}})$ is a good approximation for $S^*(\boldsymbol{\theta} + \boldsymbol{\delta}_{\text{lm}})$, and the value λ can be decreased, bringing the Levenberg–Marquardt a step closer to the Gauss–Newton step. However, if the value of Φ is small, then $\mathcal{S}(\boldsymbol{\delta}_{\text{lm}})$ is a poor estimate for $S^*(\boldsymbol{\theta} + \boldsymbol{\delta}_{\text{lm}})$, and the value of λ should be increased. The Levenberg–Marquardt algorithm for minimizing $S^*(\boldsymbol{\theta})$ is as follows (Madsen et al. 2004):

1. Set the initial value $\widehat{\boldsymbol{\theta}}_{(0)}$ for $\boldsymbol{\theta}$ and $\lambda_{(0)} = \lambda^* \max\left(\text{diag}\left(\hat{\boldsymbol{J}}_{(0)}^{\text{T}}\hat{\boldsymbol{J}}_{(0)}\right)\right)$, where λ^* is a very small and positive number.
2. For $r = 0, 1, \ldots$, calculate the following:
   $$\boldsymbol{\delta}_{\text{lm(r)}} = -\left(\hat{\boldsymbol{J}}_{(r)}^{\text{T}}\hat{\boldsymbol{J}}_{(r)} + \lambda_{(r)}\mathbf{I}_d\right)^{-1}\hat{\boldsymbol{J}}_{(r)}^{\text{T}}\boldsymbol{f}\left(\widehat{\boldsymbol{\theta}}_{(r)}\right)$$
   $$\Phi_{(r)} = \frac{S^*\left(\widehat{\boldsymbol{\theta}}_{(r)}\right) - S^*\left(\widehat{\boldsymbol{\theta}}_{(r)} + \boldsymbol{\delta}_{\text{lm(r)}}\right)}{\frac{1}{2}\boldsymbol{\delta}_{\text{lm}(r)}^{\text{T}}\left(\lambda_{(r)}\boldsymbol{\delta}_{\text{lm}(r)} - \hat{\boldsymbol{J}}_{(r)}^{\text{T}}\boldsymbol{f}\left(\widehat{\boldsymbol{\theta}}_{(r)}\right)\right)}$$
   a. If $\Phi_{(r)} > 0$, then
      $\widehat{\boldsymbol{\theta}}_{(r+1)} = \widehat{\boldsymbol{\theta}}_{(r)} + \boldsymbol{\delta}_{\text{lm}(r)}$
      $\lambda_{(r+1)} = \lambda_{(r)} \max\left(\frac{1}{3}, 1 - \left(2\Phi_{(r)} - 1\right)^3\right);\ \ell_{(r)} = 2\,.$
   b. If $\Phi_{(r)} < 0$, then
      $\widehat{\boldsymbol{\theta}}_{(r+1)} = \widehat{\boldsymbol{\theta}}_{(r)} + \boldsymbol{\delta}_{\text{lm}(r)}$
      $\lambda_{(r+1)} = \lambda_{(r)}\ell_{(r)};$
      $\ell_{(r+1)} = 2\ell_{(r)}\,.$
3. The r-th iteration stops if $\left\|\widehat{\boldsymbol{\theta}}_{(r+1)} - \widehat{\boldsymbol{\theta}}_{(r)}\right\| \leq \epsilon$, where ϵ is also positive and very small.
4. From Step 3, the algorithm produces the estimator $\widehat{\boldsymbol{\theta}}$.

To ensure convergence when estimating the parameter vector $\boldsymbol{\theta}$ in the Lorenz function $L(p; \boldsymbol{\theta})$, the initial value in the numerical process is set according to the parameter domain. For example, in the estimation of the Rohde–Lorenz function (Rohde 2009), where the parameter a_R has a domain of $(1, \infty)$, the initial value of $a_{R(0)}$ is set as $a_{R(0)} > 1$ in order to obtain the estimator $\hat{a}_R$ and achieve convergence. Conversely, convergence does not occur if the initial value of $a_{R(0)} \leq 1$.

The Amato index

The Amato index is formulated as follows (Amato 1968, Arnold 2012):

$$I_A = \int_0^1 \sqrt{1 + (L'(p; \boldsymbol{\theta}))^2}\, dp. \tag{7}$$

Because $\boldsymbol{\theta}$ is estimated by $\widehat{\boldsymbol{\theta}}$ using the nonlinear least squares method and the Levenberg–Marquardt algorithm, Eq. (7) becomes

$$\hat{I}_A = \int_0^1 \sqrt{1 + \left(L'(p; \widehat{\boldsymbol{\theta}})\right)^2}\, dp. \tag{8}$$

Based on empirical data, the Amato index is also formulated as follows (Arnold 2012):

$$\tilde{I}_A = \frac{1}{\mathcal{N}} \sum_{j=1}^{\mathcal{N}} \sqrt{1 + \left(\frac{x_j}{\bar{x}}\right)^2}\ . \tag{9}$$

The greater the Amato index value, the greater is the income or expenditure inequality in society. This index has an interval of $[\sqrt{2}, 2]$. An Amato index value of $\sqrt{2}$ signifies that the income equality is perfect, and the Lorenz curve is an egalitarian line. However, if the Amato index is equal to 2, then the income inequality is perfect.

Properties of inequality measures

Several properties are ideally inherent in a measure of inequality, although not all of them need to be satisfied. These properties, as discussed by Cowell (2011), Cowell–Flachaire (2002), and McAleer et al. (2017), are as follows:

1. Symmetry: The value of a measure of inequality should not depend on assigning a particular label to the unit entity within it.
2. Scale Independence: A measure of inequality should remain unchanged if the income of each resident or household is multiplied by a constant.
3. Non-negativity: Every measure of inequality has a value greater than or equal to zero.
4. Population Independence: The measure of inequality should not depend on the number of units or individuals in the population under consideration.
5. Egalitarian Zero: When all residents or households have the same income, the measure of inequality should be zero, indicating perfect equality. However, the Amato index, which is based on the length of the Lorenz curve, cannot reach zero when there is perfect equality, because the Lorenz curve coincides with or becomes an egalitarian line.
6. Bounded Above by Maximum Inequality: A measure of inequality should have a maximum value of one. This value reflects that there is maximum inequality (perfect inequality).

7. Transfer Principle: If income transfers occur among members of the population, the measure of inequality must necessarily change in comparison to a scenario in which no income transfers take place, and thus be sensitive to such income transfers.

Data

The employment income data per household were obtained from the Ghana Living Standards Survey IV, conducted by the Ghana Statistical Service in 1998. The survey successfully enumerated 5,999 households out of a target of 6,000. However, owing to insufficient entity records in one completed questionnaire, one household was excluded from the data set. As a result, the final data set comprises 5,998 observations.

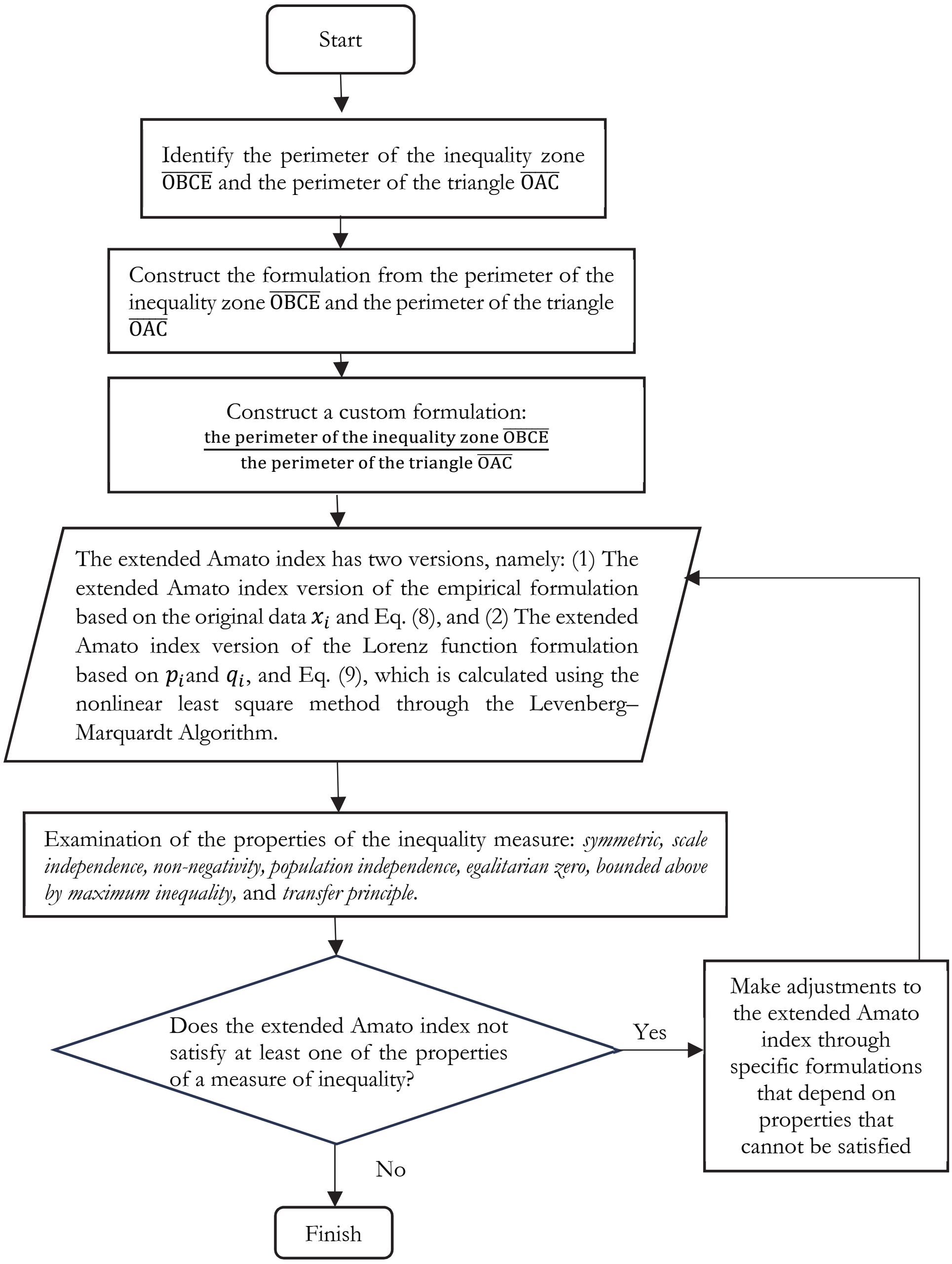
Constructing the proposed extended Amato index

The extended Amato index is constructed as follows (see Figure 2):

1. Identify the boundaries of the inequality zone $\overline{\text{OBCE}}$ and the triangle $\overline{\text{OAC}}$ to determine the perimeter of both zones.
2. Construct the general formulation of the perimeter of the inequality zone $\overline{\text{OBCE}}$ and the triangle $\overline{\text{OAC}}$, as follows:
   a. General formulation of the perimeter $\overline{\text{OBCE}}$:
   $$\rho_{\overline{\text{OBCE}}} = \mathcal{L}_{\overline{\text{OBC}}} + \mathcal{L}_{\overline{\text{OEC}}},$$
   where $\rho_{\overline{\text{OBCE}}}$ is the perimeter of $\overline{\text{OBCE}}$, $\mathcal{L}_{\overline{\text{OBC}}}$ is the length of the line $\overline{\text{OBC}}$ (the length of the egalitarian line, and $\mathcal{L}_{\overline{\text{OEC}}}$ is the length of the line $\overline{\text{OEC}}$ (the length of the Lorenz curve).
   b. General formulation of triangle perimeter $\overline{\text{OAC}}$:
   $\rho_{\overline{\text{OAC}}} = \mathcal{L}_{\overline{\text{OBC}}} + \mathcal{L}_{\overline{\text{OA}}} + \mathcal{L}_{\overline{\text{AC}}},$
   where $\rho_{\overline{\text{OAC}}}$ is the perimeter of triangle $\overline{\text{OAC}}$, $\mathcal{L}_{\overline{\text{OA}}}$ is the length of line $\overline{\text{OA}}$, and $\mathcal{L}_{\overline{\text{AC}}}$ is the length of line $\overline{\text{AC}}$.
3. Construct the formulation I_A^*, as follows:
   $$I_A^* = \frac{\rho_{\overline{\text{OBCE}}}}{\rho_{\overline{\text{OAC}}}}, \tag{10}$$
   where I_A^* is the extended Amato index.
4. From Eq. (10), we have the following:
   a. the extended Amato index based on the empirical formulation using the original data x_j and Eq. (9); and
   b. the extended Amato index based on the Lorenz function formulation using p_i and q_i (the observation sequence number and x_j must be transformed into p_i and q_i, respectively, using the cumulative proportion) and Eq. (8), which is calculated using the nonlinear least squares method and the Levenberg–Marquardt algorithm.

Figure 2

Flowchart of the extended Amato index construction method



5. Examine each of the properties of a measure of inequality (i.e. the symmetry, scale independence, non-negativity, population independence, egalitarian zero, bounded above by a maximum inequality, and transfer principle) for the extended Amato index (I_A^*) from the result of Step 4a.
6. If the extended Amato index fails to satisfy at least one of the properties in Step 5, we adjust the index. However, if it meets all the ideal properties of a measure of inequality, then we can use it to conduct the analysis.

Results

The extended Amato index

Proposition 1

The measure of inequality proposed in this study is as follows:

$$\hat{I}_A^* = \frac{\sqrt{2} + \hat{I}_A}{2 + \sqrt{2}}, \tag{11}$$

where $\hat{I}_A^*$ is the extended Amato index based on $L(p; \widehat{\boldsymbol{\theta}})$.

Proof:

Based on the information from Figure 1, Eq. (10) can be formulated as follows:

$$I_A^* = \frac{\mathcal{L}_{\overline{\mathrm{OBC}}} + \mathcal{L}_{\overline{\mathrm{OEC}}}}{\rho_{\overline{\mathrm{OAC}}}} = \frac{\mathcal{L}_{\overline{\mathrm{OBC}}} + \mathcal{L}_{L(p;\boldsymbol{\theta})}}{\rho_{\overline{\mathrm{OAC}}}}. \tag{12}$$

Furthermore, from Figure 1, we have

$$\begin{aligned} \mathcal{L}_{\overline{\mathrm{OBC}}} &= \sqrt{(\mathcal{L}_{\overline{\mathrm{OA}}})^2 + (\mathcal{L}_{\overline{\mathrm{AC}}})^2} = \sqrt{1^2 + 1^2} = \sqrt{2} \\ \rho_{\overline{\mathrm{OAC}}} &= \mathcal{L}_{\overline{\mathrm{OA}}} + \mathcal{L}_{\overline{\mathrm{AC}}} + \mathcal{L}_{\overline{\mathrm{OBC}}} \\ &= 1 + 1 + \sqrt{2} = 2 + \sqrt{2}, \end{aligned} \tag{13}$$

where $\mathcal{L}_{\overline{\mathrm{OBC}}}$ is the length of the egalitarian line $p = q$, which is also the hypotenuse of the triangle $\overline{\mathrm{OAC}}$. Next, substituting Eq. (13) into Eq. (12), we have

$$I_A^* = \frac{\sqrt{2} + \mathcal{L}_{L(p;\boldsymbol{\theta})}}{2 + \sqrt{2}}. \tag{14}$$

Because the length of the Lorenz curve $\left(\mathcal{L}_{L(p;\boldsymbol{\theta})}\right)$ is the Amato index, Eq. (14) becomes

$$\begin{aligned} I_A^* &= \frac{\sqrt{2} + I_A}{2 + \sqrt{2}} \\ &= \frac{\sqrt{2} + \int_0^1 \sqrt{1 + \left(L'(p; \boldsymbol{\theta})\right)^2}\, dp}{2 + \sqrt{2}}. \end{aligned} \tag{15}$$

Based on Eq. (15), $\boldsymbol{\theta}$ is estimated by $\widehat{\boldsymbol{\theta}}$. yielding

$$\hat{I}_A^* = \frac{\sqrt{2} + \int_0^1 \sqrt{1 + \left(L'(p;\widehat{\boldsymbol{\theta}})\right)^2}\, dp}{2+\sqrt{2}} \,. \qquad (16)$$
$$= \frac{\sqrt{2} + \hat{I}_A}{2+\sqrt{2}}.$$

Based on Eq. (16), the extended Amato index can also be formulated empirically by inserting Eq. (9) into Eq. (16), yielding

$$\tilde{I}_A^* = \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{x_i}{\bar{x}}\right)^2 + 1}}{2+\sqrt{2}}. \qquad (17)$$

Eqs. (16) and (17) are called the extended Amato index based on $L(p;\widehat{\boldsymbol{\theta}})$ and an empirical formulation, respectively. Note that when calculating $\hat{I}_A^*$, the Monte Carlo integration approach can be applied to integral components that cannot be solved analytically (Givens–Hoeting 2013).

Proposition 2

$\hat{I}_A^*$ has an upper and lower bound of [0.82843,1].

Proof:

Assuming there is perfect equality in the income distribution, meaning all households have the same income, the Lorenz curve would coincide with the egalitarian line, thus making the lengths of the lines equal. This leads to the following formula for the lower bound of $\hat{I}_A^*$ ($\ell_{\hat{I}_A^*}$):

$$\ell_{\hat{I}_A^*} = \frac{\sqrt{2}+\sqrt{2}}{2+\sqrt{2}} = \frac{2^{\frac{3}{2}}}{2+\sqrt{2}} \approx 0.82843, \qquad (18)$$

However, when there is perfect inequality in the income distribution, the Lorenz curve reaches a triangular shape $\overline{\text{OAC}}$. In this case, the perimeter of the inequality zone is equal to that of the triangle $\overline{\text{OAC}}$. This yields the following upper bound of $\hat{I}_A^*$ ($u_{\hat{I}_A^*}$):

$$u_{\hat{I}_A^*} = \frac{\sqrt{2}+2}{2+\sqrt{2}} = 1. \qquad (19)$$

Based on Eqs. (18) and (19), $\hat{I}_A^*$ has the interval [0.82843,1]. Note that when the value of $\hat{I}_A^*$ approaches one, the level of income inequality becomes more pronounced, and vice versa. However, the closer $\hat{I}_A^*$ is to 0.82843, the more the income is equally distributed.

Properties of $\tilde{I}_A^*$

In this section, we examine the properties of $\tilde{I}_A^*$ as a measure of inequality in detail. The properties are as follows:

a. Symmetric

The value of the measure of inequality should not depend on assigning a particular label to the unit entity within it.

$\tilde{I}_A^*(x_1, x_2, \dots, x_{\mathcal{N}}) = \tilde{I}_A^*(x_2, x_1, \dots, x_{\mathcal{N}}) = \tilde{I}_A^*(x_{\mathcal{N}}, x_{\mathcal{N}-1}, \dots, x_1)$

$$\tilde{I}_A^* = \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_1}{\bar{x}}\right)^2 + 1} + \sqrt{\left(\frac{x_2}{\bar{x}}\right)^2 + 1} + \cdots + \sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2 + 1}\right)}{2 + \sqrt{2}}$$

$$= \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_2}{\bar{x}}\right)^2 + 1} + \sqrt{\left(\frac{x_1}{\bar{x}}\right)^2 + 1} + \cdots + \sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2 + 1}\right)}{2 + \sqrt{2}}$$

$$\tilde{I}_A^* = \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_3}{\bar{x}}\right)^2 + 1} + \sqrt{\left(\frac{x_1}{\bar{x}}\right)^2 + 1} + \cdots + \sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2 + 1}\right)}{2 + \sqrt{2}}.$$

b. Scale independence

If each individual or household income is increased by multiplying by a constant, then the inequality does not change.

Suppose $x_i^* = \mathbb{k}x_i, \mathbb{k} > 0, \bar{x}^* = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x_i^*$;

$$\tilde{I}_A^* = \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sqrt{\left(\frac{x_i^*}{\bar{x}^*}\right)^2 + 1}}{2 + \sqrt{2}} = \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sqrt{\left(\frac{\mathbb{k}x_i}{\mathbb{k}\bar{x}}\right)^2 + 1}}{2 + \sqrt{2}}$$

$$= \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sqrt{\left(\frac{x_i}{\bar{x}}\right)^2 + 1}}{2 + \sqrt{2}}.$$

c. Non-negativity

Each measure of inequality should have a value greater than or equal to zero. Consider the formulation of $\tilde{I}_A^*$ (because X is a non-negative variable, $x_i \geq 0$). Then, $\tilde{I}_A^* > 0$.

d. Population independence

The measure of inequality does not depend on the number of basic units within it. Suppose two identical regions, each region with $\mathcal{N}$ households, are measured for income inequality (in terms of household income level) and have the same inequality value. If the two regions are combined, the result would be a total population of $2\mathcal{N}$ households. This shows that the measure of inequality does not change if there is an increase in the number of members of two identical populations.

Region 1

$$\tilde{I}^*_{A\_1} = \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{x_{1i}}{\bar{x}_1}\right)^2 + 1}}{2 + \sqrt{2}}.$$

Region 2

$$\tilde{I}^*_{A\_2} = \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{x_{2i}}{\bar{x}_2}\right)^2 + 1}}{2 + \sqrt{2}}.$$

Regions 1 and 2 are assumed to be identical, and when merged, $x_{1i} = x_{2i}, \bar{x}_1 = \bar{x}_2$,

$$\tilde{I}^*_{A\_12} = \frac{\sqrt{2} + \frac{1}{2\mathcal{N}} \sum_{i=1}^{2\mathcal{N}} \sqrt{\left(\frac{x_i^*}{\bar{x}^*}\right)^2 + 1}}{2 + \sqrt{2}}.$$

Because they are identical, in Region 1, each unit has an income of

$x_i^* = x_{1i} + x_{2i} = 2x_{1i} = 2x_{2i}$, yielding an average of $\bar{x}^* = \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} 2x_{1i} = 2\bar{x}_1$, and

$$\tilde{I}^*_{A\_12} = \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{2x_{1i}}{\bar{x}^*}\right)^2 + 1}}{2 + \sqrt{2}}$$

$$= \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{2x_{1i}}{2\bar{x}_1}\right)^2 + 1}}{2 + \sqrt{2}}$$

$$= \frac{\sqrt{2} + \frac{1}{\mathcal{N}} \sum_{i=1}^{\mathcal{N}} \sqrt{\left(\frac{x_{1i}}{\bar{x}_1}\right)^2 + 1}}{2 + \sqrt{2}} = \tilde{I}^*_{A\_1}.$$

For example, suppose there are two regions, namely, Region 1 and Region 2, in which households have identical incomes (in USD), as follows:

Region 1: {10,000; 5,000; 3,000; 4,000; 20,000; 1,000}

Region 2: {10,000; 5,000; 3,000; 4,000; 20,000; 1,000}

If the two regions are merged, they become

Region 1 and 2 merged: {10,000; 10,000; 5,000; 5,000; 3,000; 3,000; 4,000; 4,000; 20,000; 20,000; 1,000; 1,000}.

Therefore, based on Eq. (17), the extended Amato indices for Regions 1 and 2 and for both regions combined produce the same value of 0.86036 (see Table 4).

Table 4

Hypothetical example for proving the property of population independence

	Region 1		Region 2		Region 1 and Region 2 are merged	
	x_{1i}	$\sqrt{\left(\frac{x_{1i}}{\bar{x}_1}\right)^2+1}$	x_{2i}	$\sqrt{\left(\frac{x_{2i}}{\bar{x}_2}\right)^2+1}$	x_i^*	$\sqrt{\left(\frac{x_i^*}{\bar{x}^*}\right)^2+1}$
	10,000	1.71668	10,000	1.71668	10,000	1.71668
	5,000	1.21932	5,000	1.21932	10,000	1.71668
	3,000	1.08408	3,000	1.08408	5,000	1.21932
	4,000	1.14522	4,000	1.14522	5,000	1.21932
	20,000	2.96446	20,000	2.96446	3,000	1.08408
	1,000	1.00969	1,000	1.00969	3,000	1.08408
					4,000	1.14522
					4,000	1.14522
					20,000	2.96446
					20,000	2.96446
					1,000	1.00969
					1,000	1.00969
Total	43,000	9.13945	43,000	9.13945	86,000	18.27889
Average	7,167	1.52324	7,167	1.52324	7,167	1.52324
Number of observations ($\mathcal{N}$)	6		6		12	
$\tilde{I}_A^*$	0.86036		0.86036		0.86036	

Table 4 offers information about the computation of the extended Amato index for two regions with identical household incomes, both separately and when merged. The extended Amato index retains the same value, even after merging the two regions, indicating that the number of observations (i.e. the population) does not affect the extended Amato index.

e. Egalitarian zero

If all households have the same income, then the income inequality is zero. Suppose $x_i = x, \bar{x} = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x = x$, and

$$\tilde{I}_A^* = \frac{\sqrt{2} + \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sqrt{\left(\frac{x}{x}\right)^2+1}}{2+\sqrt{2}} = \frac{\sqrt{2}+\sqrt{2}}{2+\sqrt{2}} = \frac{2\sqrt{2}}{2+\sqrt{2}} \quad \rightarrow \quad \tilde{I}_A^* \neq 0.$$

In this context, $\tilde{I}_A^*$ does not satisfy the egalitarian zero property. The inequality zone coincides with the egalitarian line when the household incomes are equal. Hence, the component of the remaining perimeter of the inequality zone is simply the length of the egalitarian line, and thus $\tilde{I}_A^*$ will never be zero.

f. Bounded above by maximum inequality

When a measure of inequality has a maximum value of one, we have maximum inequality (perfect inequality). In this case, the Lorenz curve forms the shape of a triangle $\overline{\text{OAC}}$, where the perimeter of the inequality zone $\overline{\text{OBCE}}$ becomes equal to that of the triangle $\overline{\mathrm{OAC}}$. This makes I_A (the length of the Lorenz curve) equal to two, and thus

$$\tilde{I}_A^* = \frac{\sqrt{2}+I_A}{2+\sqrt{2}} = \frac{\sqrt{2}+2}{2+\sqrt{2}} = 1.$$

g. Transfer principle

If income transfers occur among members of the population, the measure of inequality must necessarily change in comparison to a scenario in which no income transfers take place, and thus be sensitive when such transfers occur. For example, suppose in a region that out of $\mathcal{N}$ households, only the first household transfers its income of Υ $(\Upsilon > 0)$ to the second household. Here, the income of the first individual is $x_1 - \Upsilon$ and that of the second is $x_2 + \Upsilon$. Thus, the inequality before and after the transfer must differ.

– Extended Amato index before transfer:

$$\tilde{I}_A^* = \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}}\sqrt{\left(\frac{x_i}{\bar{x}}\right)^2+1}}{2+\sqrt{2}}$$

$$= \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_1}{\bar{x}}\right)^2+1}+\sqrt{\left(\frac{x_2}{\bar{x}}\right)^2+1}+\cdots+\sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2+1}\right)}{2+\sqrt{2}}, \bar{x} = \frac{1}{\mathcal{N}}\sum_{i=1}^{\mathcal{N}} x_i\,.$$

– Extended Amato index after transfer:

$$\tilde{I}_A^{**} = \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_1-\Upsilon}{\bar{x}^*}\right)^2+1}+\sqrt{\left(\frac{x_2+\Upsilon}{\bar{x}^*}\right)^2+1}+\cdots+\sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}^*}\right)^2+1}\right)}{2+\sqrt{2}}$$

$$\bar{x}^* = \frac{1}{\mathcal{N}}\left((x_1-\Upsilon)+(x_2+\Upsilon)+x_3+\cdots+x_{\mathcal{N}}\right) = \bar{x}$$

$$\tilde{I}_A^{**} = \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\left(\sqrt{\left(\frac{x_1-\Upsilon}{\bar{x}}\right)^2+1}+\sqrt{\left(\frac{x_2+\Upsilon}{\bar{x}}\right)^2+1}+\cdots+\sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2+1}\right)}{2+\sqrt{2}}$$

$$= \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\left(\sqrt{\frac{x_1^2-2x_1\Upsilon+\Upsilon^2}{\bar{x}^2}+1}+\sqrt{\frac{x_2^2+2x_2\Upsilon+\Upsilon^2}{\bar{x}^2}+1}+\cdots+\sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2+1}\right)}{2+\sqrt{2}}$$

$$= \frac{\sqrt{2}+\frac{1}{\mathcal{N}}\left(\sqrt{\frac{x_1^2}{\bar{x}^2}+1+\left(\frac{-2x_1\Upsilon+\Upsilon^2}{\bar{x}^2}\right)}+\sqrt{\frac{x_2^2}{\bar{x}^2}+1+\left(\frac{2x_2\Upsilon+\Upsilon^2}{\bar{x}^2}\right)}+\cdots+\sqrt{\left(\frac{x_{\mathcal{N}}}{\bar{x}}\right)^2+1}\right)}{2+\sqrt{2}}.$$

The value of the inequality index changes due to transfer, $I_A^* \neq I_A^{**}$.

The adjusted extended Amato index

The extended Amato index did not fulfil the egalitarian zero property, because of the perimeter concept that includes the length of the Lorenz curve, which cannot be zero. In order to address this limitation, the extended Amato index can be scaled into the domain of values between zero and one. The scaling process assigns a lower limit of zero and an upper limit of one to the index, ensuring a standardized scale. In this study, we apply minimum-maximum scaling. The adjusted extended Amato index $I_{A_n}^*$ is calculated as follows:

$$I_{A_n}^* = \frac{I_A^* - \frac{2^{\frac{3}{2}}}{2+\sqrt{2}}}{1 - \frac{2^{\frac{3}{2}}}{2+\sqrt{2}}} = \left(3 + 2^{\frac{3}{2}}\right) I_A^* - 2\left(1+\sqrt{2}\right)$$

$$\hat{I}_{A_n}^* = \left(3 + 2^{\frac{3}{2}}\right) \hat{I}_A^* - 2\left(1+\sqrt{2}\right) \approx 5.82843\hat{I}_A^* - 4.82843 \quad (20)$$

$$\tilde{I}_{A_n}^* = \left(3 + 2^{\frac{3}{2}}\right) \tilde{I}_A^* - 2\left(1+\sqrt{2}\right) \approx 5.82843\tilde{I}_A^* - 4.82843. \quad (21)$$

Eqs. (20) and (21) use the extended Amato index based on $L(p;\widehat{\boldsymbol{\theta}})$ and on the empirical formulation, respectively. Because $I_{A_n}^*$ is derived from $\hat{I}_A^*$, which satisfies all properties of the inequality measurement index except egalitarian zero, we construct $I_{A_n}^*$ to overcome the weakness of the egalitarian zero in $\hat{I}_A^*$ and, thus, $\hat{I}_{A_n}^*$ satisfies all of the aforementioned properties of an inequality measurement index.

Empirical study

By constructing an empirical Lorenz curve, the cumulative proportion mechanism transforms the income employment per household. The candidate Lorenz functions used in this study are as follows:

1. The Lorenz–Hossain and Saeki (Lorenz–HS) function

The Lorenz–HS function is specified in Eq. (22) (Hossain–Saeki 2003):

$$L_{HS}(p;\vartheta_1,\vartheta_2,\vartheta_3) = p^{\vartheta_1} e^{\vartheta_2(p-1)}\left(1-(1-p)^{\vartheta_3}\right). \quad (22)$$

This function contains three parameters, $\boldsymbol{\theta}_{HS} = (\vartheta_1\ \vartheta_2\ \vartheta_3)^{\mathrm{T}}$, which have domains $\vartheta_1 \geq 0; \vartheta_2 \geq 0;$ and $0 < \sigma_3 \leq 1,$ respectively. The Amato index derived from $L_{HS}(p;\vartheta_1,\vartheta_2,\vartheta_3)$ using Eqs. (1) and (2) cannot be solved analytically, but the solution can be solved by using a Monte Carlo approach. In addition, because $\boldsymbol{\theta}_{HS}$ is estimated by $\widehat{\boldsymbol{\theta}}_{HS} = \left(\hat{\vartheta}_1\ \hat{\vartheta}_2\ \hat{\vartheta}_3\right)^{\mathrm{T}}$, the Amato index is formulated as follows:

$$\hat{I}_{A-HS} \approx \frac{1}{\mathcal{M}} \sum_{\iota=1}^{\mathcal{M}} \mathcal{I}_{HS}(z_\iota)\,, \iota = 1, \ldots, \mathcal{M}\,, \quad (23)$$

where

$$\mathcal{I}_{HS}(z_\iota) = (1 + (\hat{\vartheta}_1 e^{\hat{\vartheta}_2(z_\iota - 1)}(1 - (1 - z_\iota)^{\hat{\vartheta}_3})z_\iota^{\hat{\vartheta}_1 - 1} + \hat{\vartheta}_2 e^{\hat{\vartheta}_2(z_\iota - 1)}(1 - (1 - z_\iota)^{\hat{\vartheta}_3})z_\iota^{\hat{\vartheta}_1} + \hat{\vartheta}_3 e^{\hat{\vartheta}_2(z_\iota - 1)}(1 - z_\iota)^{\hat{\vartheta}_3 - 1} z_\iota^{\hat{\vartheta}_1})^2)^{1/2}.$$

Here, z_ι is generated from the uniform distribution (0,1). Accordingly, the generated units of z_ι are represented by $\mathcal{M}$; in this case, $\mathcal{M}$ is a large number.

Based on Eq. (23), the adjusted extended Amato index is formulated as follows:

$$\hat{I}^*_{A-HS_n} = \left(3 + 2^{\frac{3}{2}}\right)\hat{I}^*_{A-HS} - 2\left(1 + \sqrt{2}\right),\ \ \hat{I}^*_{A-HS} = \frac{\sqrt{2} + \hat{I}_{A-HS}}{2 + \sqrt{2}}.$$

2. The Lorenz–Kakwani function

The Lorenz–Kakwani function is specified in Eq. (24) (Kakwani 1980):

$$L_{\mathcal{K}}(p; \delta_1, \delta_2, \delta_3) = p - \delta_1 p^{\delta_2}(1 - p)^{\delta_3}. \tag{24}$$

This function contains three parameters, $\boldsymbol{\theta}_{\mathcal{K}} = (\delta_1\ \delta_2\ \delta_3)^{\mathrm{T}}$, with domains $0 < \delta_1 \leq 1; \delta_2 \geq 1;$ and $0 < \delta_3 \leq 1,$ respectively. Furthermore, recall that the Amato index derived from $L_{\mathcal{K}}(p; \delta_1, \delta_2, \delta_3)$ using Eqs. (1) and (2) cannot be solved analytically, but can be solved using a Monte Carlo approach. Furthermore, because $\boldsymbol{\theta}_{\mathcal{K}}$ is estimated by $\widehat{\boldsymbol{\theta}}_{\mathcal{K}} = \left(\hat{\delta}_1\ \hat{\delta}_2\ \hat{\delta}_3\right)^{\mathrm{T}}$, the Amato index is formulated as follows:

$$\hat{I}_{A-\mathcal{K}} \approx \frac{1}{\mathcal{M}} \sum_{\iota=1}^{\mathcal{M}} \mathcal{I}_{\mathcal{K}}(z_\iota),\ \iota = 1, \ldots, \mathcal{M}, \tag{25}$$

where

$$\mathcal{I}_{\mathcal{K}}(z_\iota) = \left(1 + \left(1 - \hat{\delta}_1\hat{\delta}_2(1 - z_\iota)^{\hat{\delta}_3} z_\iota^{\hat{\delta}_2 - 1} + \hat{\delta}_1\hat{\delta}_3(1 - z_\iota)^{\hat{\delta}_3 - 1} z_\iota^{\hat{\delta}_2}\right)^2\right)^{1/2}.$$

Based on Eq. (25), the adjusted extended Amato index is formulated as follows:

$$\hat{I}^*_{A-\mathcal{K}_n} = \left(3 + 2^{\frac{3}{2}}\right)\hat{I}^*_{A-\mathcal{K}} - 2\left(1 + \sqrt{2}\right),\ \ \hat{I}^*_{A-\mathcal{K}} = \frac{\sqrt{2} + \hat{I}_{A-\mathcal{K}}}{2 + \sqrt{2}}.$$

Table 5

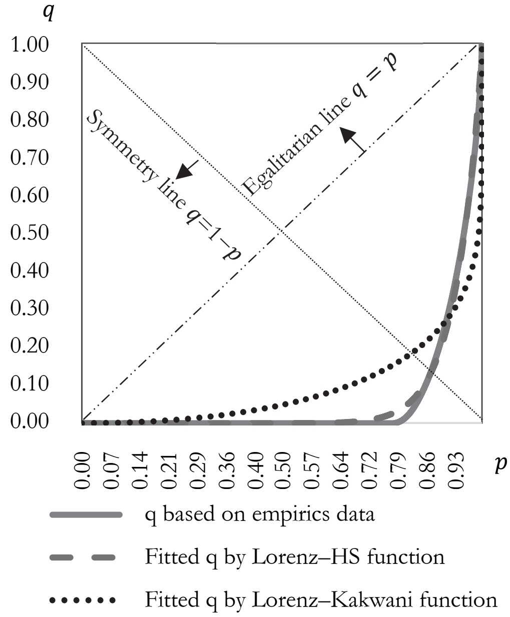
Parameter estimation results of the Lorenz–HS function and Lorenz–Kakwani function

Lorenz–HS function		Lorenz–Kakwani function	
estimator	value	estimator	value
$\hat{\vartheta}_1$	12.67454	$\hat{\delta}_1$	1.00000
$\hat{\vartheta}_2$	0.00000	$\hat{\delta}_2$	1.00000
$\hat{\vartheta}_3$	0.66796	$\hat{\delta}_3$	0.14569
MSE	**1.23662 × 10^{-4}**	MSE	8.24736 × 10^{-3}
MAE	**5.67930 × 10^{-3}**	MAE	6.28036 × 10^{-2}

Note: MSE: mean squared error and MAE: mean absolute error.

As discussed earlier, we use the nonlinear least squares method and the Levenberg–Marquardt algorithm to estimate the parameters of both the Lorenz–HS and the Lorenz–Kakwani functions when fitting Ghana's empirical Lorenz curve based on income employment per household. The parameter estimation results for both functions are presented in Table 5.

Figure 3
Visualization of fitting the Lorenz–HS function and Lorenz–Kakwani function to Ghana's empirical Lorenz curve in 1998



The findings, presented in Table 5, indicate that the Lorenz–HS function outperforms the Lorenz–Kakwani function in terms of fitting capability. This conclusion is supported by the lower mean squared error (MSE) and mean absolute error (MAE) values obtained from the Lorenz–HS function than those from the Lorenz–Kakwani function. Furthermore, Figure 3 illustrates that the Lorenz–HS function better fits the empirical Lorenz curve than the Lorenz–Kakwani function does. The inequality zone formed by the empirical Lorenz curve and the egalitarian line is close to the inequality triangle. This condition indicates that income inequality is very high, and that the Lorenz–HS function forms a similar zone of inequality to that of the Lorenz–Kakwani function.

From Table 6, the empirical Amato index, empirical extended Amato index, and empirical adjusted extended Amato index values are 1.82728, 0.94941, and 0.70515, respectively. Interpreting the adjusted extended Amato index values using Oshima's range (Oshima 1976), we can conclude that there was significant income inequality in Ghanaian employment in 1998. This observation aligns with the substantial gap between the inequality zone and the egalitarian line, as depicted in Figure 3. The empirical Lorenz curve exhibits asymmetry, with a bulge at the bottom left of the inequality zone, indicating that the bottom 50% income group contributes significantly to the income inequality.

Table 6

Estimated values of the Amato index, extended Amato index, and adjusted extended Amato index based on empirical formulation, Lorenz–HS function, and Lorenz–Kakwani function

	Empirics	HS–Lorenz	Kakwani–Lorenz
The Amato index	1.82728	1.80689	1.74972
The extended Amato index	0.94941	0.94344	0.92670
The adjusted extended Amato index	0.70515	0.67034	0.57275

When comparing the results from the two functions, it is evident that the Lorenz–HS function produces values for the Amato index, extended Amato index, and adjusted extended Amato index that are more closely aligned with the empirical Amato index and its progression than those produced by the Lorenz–Kakwani function. This result can be ascribed to the Lorenz–HS function's superior fit with the empirical Lorenz curve, as depicted in Figure 3. The precise specification of the Lorenz function gains considerable credibility, owing to its strong alignment with the empirical Lorenz curve.

Local authorities can use the following strategies to reduce income inequality:

a. *Enhance the minimum wage*: Increasing the minimum wage can contribute to mitigating income inequality by ensuring that low-income workers earn a wage that sustains their livelihood (Powell 2014). This can also help to reduce poverty and increase economic mobility.
b. *Extend the scope of the earned income tax credit (EITC)*: The EITC is a tax credit designated for workers with low to moderate income, and can assist in curtailing poverty and fostering upward economic mobility. Broadening the EITC's coverage would help to decrease income inequality by providing supplementary assistance to modest-income workers (Powell 2014).
c. *Foster asset accumulation for working households*: Policy measures that facilitate the accumulation of assets, such as savings accounts or homeownership opportunities, can play a role in diminishing income inequality by fostering economic mobility among working families (Powell 2014).
d. *Allocate resources to education*: Education is pivotal to increasing economic mobility and reducing income inequality. Policies that invest in education, such as augmenting funding for public schools or offering tuition-free college education, can curtail income inequality by providing greater avenues of income for low-income individuals, thus enhancing their economic prospects.
e. *Enhance tax system progressivity*: A progressive tax system in which higher-income individuals contribute a greater proportion of their earnings as taxes, can reduce income inequality through wealth redistribution (Dabla-Norris et al.

2015). Such progressivity can be accomplished by increasing tax rates for high-earning individuals and closing tax loopholes that favour the affluent.

f. *Ensure equitable opportunities*: Policies that promote equal opportunities, such as eliminating discriminatory regulations and practices, and investing in skills development, can reduce income inequality by guaranteeing an equitable platform for all individuals to attain success.

Significantly, policies aimed at diminishing income inequality should be motivated not only by the intention to improve social outcomes, but also to maintain long-term economic growth (Cingano 2014). By enacting such policies, local authorities can work towards reducing income inequality, while simultaneously fostering economic mobility for all members of society.

Conclusions

We derive an alternative inequality index as the ratio of the perimeter of the inequality zone perimeter ($\overline{\mathrm{OBCE}}$) to the perimeter of the triangle $\overline{\mathrm{OAC}}$, incorporating Amato's index. The extended Amato index developed in this study satisfies all fundamental properties of an inequality index, except the egalitarian zero condition. Thus, we applied a scaling process to obtain the adjusted extended Amato index, which does satisfy all of these properties (symmetry, scale independence, non-negativity, population independence, the egalitarian zero condition, bounded above by maximum inequality, and the transfer principle). Notably, the extended Amato index is consistent with the Lorenz curve. Nonetheless, ensuring the accuracy and credibility of the inequality index values is crucial. This can be achieved by selecting a Lorenz function that adequately fits the empirical Lorenz curve.

An empirical study, using total income employment data from Ghana, reveals a high level of income inequality among households. This is evidenced by the extended Amato index and adjusted extended Amato index values of 0.94941 and 0.70515, respectively, both of which fall into the high inequality category. To address this elevated level of inequality, local authorities could implement strategies such as increasing the minimum wage, expanding the EITC, promoting asset accumulation for working households, investing in education, enhancing the progressivity of the tax system, and ensuring equitable opportunities. The goal of these policies is to reduce income inequality and enhance economic mobility, thereby improving social outcomes and sustaining long-term economic growth.

Determining the Lorenz function involves selecting the best specification from the proposed candidates, based on which yields the lowest values for goodness-of-fit measures, such as the MAE and MSE. However, this method lacks a probabilistic mechanism, potentially leading to coincidental model selection, owing to its single-trial nature. Consequently, future research should include a probabilistic mechanism for specifying the Lorenz function. This could be achieved using methods such as the bootstrap approach (MacKinnon 2007, Smaga 2017), or by forming posterior probabilities using the Monte Carlo Markov chain process, as demonstrated by Carlin–Chib (1995).

REFERENCES

AGGARWAL, V. (1984): On optimum aggregation of income distribution data *Sankhyā: The Indian Journal of Statistics, Series B* 46 (3): 343–355. https://doi.org/10.2307/25052360

AMATO, V. (1968): *Metodologia Statistica Strutturale* (Vol. 1). CACUCCI, Bari.

ARNOLD, B. C. (2012): On the Amato inequality index *Statistics and Probability Letters* 82 (8): 1504–1506. https://doi.org/10.1016/j.spl.2012.04.020

BANERJEE, S.–CHAKRABARTI, B. K.–MITRA, M.–MUTUSWAMI, S. (2019): On the Kolkata index as a measure of income inequality *Physica A: Statistical Mechanics and Its Applications* 545: 123178. https://doi.org/10.1016/j.physa.2019.123178

BANERJEE, S.–CHAKRABARTI, B. K.–MITRA, M.–MUTUSWAMI, S. (2020): Inequality measures: The Kolkata index in comparison with other measures *Frontier in Physics* 8: 562182. https://doi.org/10.3389/fphy.2020.562182

BASMANN, R. L.–HAYES, K. J.–SLOTTJE, D. J.–JOHNSON, J. D. (1990): A general functional form for approximating the Lorenz curve *Journal of Econometrics* 43 (1–2): 77–90. https://doi.org/10.1016/0304-4076(90)90108-6

CARLIN, B. P.–CHIB, S. (1995): Bayesian model choice via Markov chain Monte Carlo methods *Journal of the Royal Statistical Society: Series B (Methodological)* 57 (3): 473–484. https://doi.org/10.1111/j.2517-6161.1995.tb02042.x

CHANG, R. K. R.–HALFON, N. (1997): Geographic distribution of pediatricians in the United States: An analysis of the fifty states and Washington, DC. *Pediatrics* 100 (2): 172–179. https://doi.org/10.1542/peds.100.2.172

CHATTERJEE, A.–GHOSH, A.–CHAKRABARTI, B. K. (2017): Socio-economic inequality: Relationship between Gini and Kolkata indices *Physica A: Statistical Mechanics and Its Applications* 466: 583–595. https://doi.org/10.1016/j.physa.2016.09.027

CHOTIKAPANICH, D. (1993): A comparison of alternative functional forms for the Lorenz curve *Economics Letters* 41 (2): 129–138. https://doi.org/10.1016/0165-1765(93)90186-g

CINGANO, F. (2014): Trends in income inequality and its impact on economic growth *OECD Social, Employment and Migration Working Papers* No. 163., OECD Publishing, Paris. http://dx.doi.org/10.1787/5jxrjncwxv6j-en

COWELL, F. A. (2011): *Measuring Inequality* (Third edition) Oxford University Press, Oxford.

DAMGAARD, C.–WEINER, J. (2000): Describing inequality in plant size or fecundity *Ecology* 81 (4): 1139–1142. https://doi.org/10.1890/0012-9658(2000)081[1139:DIIPSO]2.0.CO;2

DELBOSC, A.–CURRIE, G. (2011): Using Lorenz curves to assess public transport equity *Journal of Transport Geography* 19 (6): 1252–1259. https://doi.org/10.1016/j.jtrangeo.2011.02.008

FOSTER, J. E.–SHORROCKS, A. F. (1988): Poverty orderings *Econometrica* 56 (1): 173–177. https://doi.org/10.2307/1911846

GIVENS, G. H.–HOETING, J. A. (2013): *Computational Statistics* (Second edition) John Wiley & Sons, Inc., New Jersey.

GRESELIN, F.–PASQUAZZI, L.–ZITIKIS, R. (2010): Zenga's new index of economic inequality, its estimation, and an analysis of incomes in Italy *Journal of Probability and Statistics* 2010: 718905. https://doi.org/10.1155/2010/718905

GUPTA, M. R. (1984): Functional form for estimating the Lorenz curve *Econometrica* 52 (5): 1313–1314. https://doi.org/10.2307/1911001

HARVEY, D. J.–GANGE, A. C.–HAWES, C. J.–RINK, M. (2011): Bionomics and distribution of the stag beetle, Lucanus cervus (L.) across Europe *Insect Conservation and Diversity* 4 (1): 23–38. http://dx.doi.org/10.1111/j.1752-4598.2010.00107.x

HOSSAIN, S.–SAEKI, C. (2003): A New functional form for estimating Lorenz curves *Journal of Business and Economics Research* 1 (5): 81–94. https://doi.org/https://doi.org/10.19030/jber.v1i5.3014

KAKWANI, N. (1980): On a class of poverty measures *Econometrica* 48 (2): 437–446. https://doi.org/10.2307/1911106

KAKWANI, N. C.–PODDER, N. (1973): On the estimation of Lorenz curves for grouped observations *International Economic Review* 14 (2): 278–292. https://doi.org/10.2307/2525921

KASHOUR, M. (2023): Interlinkages between human development, residential energy consumption, and energy efficiency for the EU-27 Member States, 2010–2018. *Regional Statistics* 13 (1): 36–54. https://doi.org/10.15196/RS130102

KOBAYASHI, Y.–TAKAKI, H. (1992): Geographic distribution of physicians in Japan *The Lancet* 340 (8832): 1391–1393. https://doi.org/10.1016/0140-6736(92)92569-2

KOCZISZKY, G.–BENEDEK, J.–SZENDI, D. (2018): The impact of the 2008 financial crisis on household income and wealth in Visegrad countries *Regional Statistics* 8 (1): 141–167. https://doi.org/10.15196/RS080102

LANGEL, M.–TILLÉ, Y. (2012): Inference by linearization for Zenga's new inequality index: A comparison with the Gini index *Metrika* 75 (8): 1093–1110. https://doi.org/10.1007/s00184-011-0369-1

LEE, W.-C. (1999): Probabilistic analysis of global performances of diagnostic tests: Interpreting the Lorenz curve-based summary measures *Statistics in Medicine* 18 (4): 455–471. https://doi.org/10.1002/(SICI)1097-0258(19990228)18:4%3C455::AID-SIM44%3E3.0.CO;2-A

LORENZ, M. O. (1905): Methods of measuring the concentration of wealth *American Statistical Association* 9 (70): 209–219. https://doi.org/10.2307/2276207

LU, H. L.–LI, F. F.–ZHAO, K.–WANG, H. R.–WANG, G. Q.–QIU, J. (2021): Indices for exploring information in Lorentz curve of daily precipitation and their application in natural disaster risk assessment *Journal of Hydrology* 603 (Part A): 126840. https://doi.org/10.1016/j.jhydrol.2021.126840

MacKinnon, J. G. (2007): Bootstrap hypothesis testing. In: Belsley, D. A.–Kontoghiorghes, E. J. (eds.): *Handbook of computational econometrics* pp. 183–213., John Wiley & Sons, Ltd., Hoboken, New Jersey. https://doi.org/10.1002/9780470748916.ch6

Ortega, P.–Martin, G.–Fernandez, A.–Ladoux, M.–Garcia, A. (1991): A new functional form for estimating Lorenz curves *Review of Income and Wealth Series* 37 (4): 447–452. https://doi.org/10.1111/j.1475-4991.1991.tb00383.x

Oshima, H. T. (1976): A need for an integrated statistical system for manpower planning in Southeast Asia *EKI* 24 (1): 71–84.

Paul, S.–Shankar, S. (2020): An alternative single parameter functional form for Lorenz curve *Empirical Economics* 59 (3): 1393–1402. https://doi.org/10.1007/s00181-019-01715-3

Pietra, G. (1915): Delle Relazioni tra gli Indici di Variabilita. In: *Atti Del Reale Istituto Veneto Di Scienze, Serie 8., tomo 74.,* Scienze matematiche e naturali (1914–1915). pp. 775–804., Presso la Segreteria dell'Istituto, Venezia.

Rasche, R. H.–Gaffney, J.–Koo, A. Y. C.–Obst, N. (1980): Functional forms for estimating the Lorenz curve *Econometrica* 48 (4): 1061–1062. https://doi.org/10.2307/1912948

Raza, A.–Akuh, R.–Safdar, M.–Zhong, M. (2023): Public transport equity with the concept of time-dependent accessibility using Geostatistics methods, Lorenz curves, and Gini coefficients *Case Studies on Transport Policy* 11: 100956. https://doi.org/10.1016/j.cstp.2023.100956

Rohde, N. (2009): An alternative functional form for estimating the Lorenz curve *Economics Letters* 105 (1): 61–63. https://doi.org/10.1016/j.econlet.2009.05.015

Sadras, V.–Bongiovanni, R. (2004): Use of Lorenz curves and Gini coefficients to assess yield inequality within paddocks *Field Crops Research* 90 (2–3): 303–310. https://doi.org/10.1016/j.fcr.2004.04.003

Sarabia, J. M. (1997): A hierarchy of Lorenz curves based on the generalized Tukey's Lambda distribution *Econometric Reviews* 16 (3): 305–320. https://doi.org/10.1080/07474939708800389

Sarabia, J. M.–Prieto, F.–Jordá, V. (2015): About the hyperbolic Lorenz curve *Economics Letters* 136: 42–45. https://doi.org/10.1016/j.econlet.2015.09.005

Sarabia, J. M.–Prieto, F.–Sarabia, M. (2010): Revisiting a functional form for the Lorenz curve *Economics Letters* 107 (2): 249–252. https://doi.org/10.1016/j.econlet.2010.01.038

Sarabia, J.-M.–Castillo, E.–Slottje, D. J. (2001): An exponential family of Lorenz curves *Source: Southern Economic Journal* 67 (3): 748–756. https://doi.org/10.2307/1061463

Schader, M.–Schmid, F. (1994): Fitting parametric Lorenz curves to grouped income distributions – A critical note *Empirical Economics* 19 (3): 361–370. https://doi.org/10.1007/bf01205943

Sitthiyot, T.–Holasut, K. (2020): A simple method for measuring inequality *Palgrave Communications* 6: 112. https://doi.org/10.1057/s41599-020-0484-6

Slottje, D. (1989): *The structure of earnings and the measurement of income inequality in the U.S.* (1st ed.) North-Holland, Amsterdam.

SMAGA, Ł. (2017): Bootstrap methods for multivariate hypothesis testing *Communications in Statistics – Simulation and Computation* 46 (10): 7654–7667. https://doi.org/10.1080/03610918.2016.1248573

SMITH, G. C. (1947): Lorenz curve analysis of industrial decentralization *Journal of the American Statistical Association* 42 (240): 591–596. https://doi.org/10.1080/01621459.1947.10501951

SZÉP, T.–TÓTH, G.–LABELLE, M. C. (2022): Farewell to the European Union's east-west divide: Decoupling energy lifts the well-being of households, 2000–2018. *Regional Statistics* 12 (3): 159–190. https://doi.org/10.15196/RS120307

WANG, Z.–SMYTH, R.–NG, Y. K. (2009): A new ordered family of Lorenz curves with an application to measuring income inequality and poverty in rural China *China Economic Review* 20 (2): 218–235. https://doi.org/10.1016/j.chieco.2008.12.003

ZENGA, M. (2007): Inequality curve and inequality index based on the ratios between lower and upper arithmetic means *Statistica & Applicazioni* 5 (1): 3–27.

INTERNET SOURCES

COWELL, F. A.–FLACHAIRE, E. (2002): Sensitivity of inequality measures to extreme values *STICERD – Distributional Analysis Research Programme Papers* 60. http://sticerd.lse.ac.uk/research/darp.asp (downloaded: May 2023)

DABLA-NORRIS, E.–KOCHHAR, K.–SUPHAPHIPHAT, N.–RICKA, F.–TSOUNTA, E. (2015): Causes and consequences of income inequality: A global perspective *IMF Staff Discussion Note* SDN/15/13. https://www.imf.org/external/pubs/ft/sdn/2015/sdn1513.pdf (downloaded: May 2023)

MADSEN, K.–NIELSEN, H. B.–TINGLEFF, O. (2004): *Methods for Non-linear Least Squares Problems* (2nd ed.). Informatics and Mathematical Modelling, Technical University of Denmark, Lyngby. http://www2.imm.dtu.dk/pubdb/edoc/imm3215.pdf (downloaded: May 2023)

MCALEER, M.–RYU, H. K.–SLOTTJE, D. J. (2017): A New inequality measure that is sensitive to extreme values and asymmetries *Tinbergen Institute Discussion Paper* No. 17–102/III, Tinbergen Institute, Amsterdam and Rotterdam. https://www.econstor.eu/bitstream/10419/177670/1/17102.pdf (downloaded: May 2023)

POWELL, J. A. (2014): *Six policies to reduce economic inequality* https://belonging.berkeley.edu/six-policies-reduce-economic-inequality (downloaded: May 2023)